

# ON IMPROVEMENT OF YOUNG INEQUALITY USING THE KONTROVICH CONSTANT

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**ABSTRACT.** Some improvements of Young inequality and its reverse for positive numbers with Kontrovich constant are given. Using these inequalities some operator versions and Hilbert-Schmidt norm versions for matrices are proved.

## 1. INTRODUCTION

Let  $a, b$  be two positive number. The famous Young inequality states that

$$a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b,$$

for every  $0 \leq \nu \leq 1$ . By defining weighted arithmetic and geometric means as

$$a\nabla b = (1-\nu)a + \nu b, \quad a\sharp_\nu b = a^{1-\nu}b^\nu,$$

we can consider the Young inequality as weighted arithmetic-geometric means inequality. This inequality has received an increasing attention in the literature.

One of the best improvement of Young inequality, was obtained by F. Kittaneh and Y. Manasrah [7], as follows:

$$a\sharp_\nu b + r(\sqrt{a} - \sqrt{b})^2 \leq a\nabla_\nu b$$

where  $r = \min\{\nu, 1-\nu\}$  and  $s = \max\{\nu, 1-\nu\}$ .

The authors of [6] obtained another refinement of the Young inequality as follows:

$$(a\sharp_\nu b)^2 + r^2(a-b)^2 \leq (a\nabla_\nu b)^2,$$

where  $r = \min\{\nu, 1-\nu\}$ .

In [14], the authors obtained another improvement of the Young inequality and its reverse as follows:

$$K(\sqrt{h}, 2)^{r'} a\sharp_\nu b \leq a\nabla_\nu b - r(\sqrt{a} - \sqrt{b})^2, \quad (1.1)$$

and

$$a\nabla_\nu b - R(\sqrt{a} - \sqrt{b})^2 \leq K(\sqrt{h}, 2)^{-r'} a\sharp_\nu b, \quad (1.2)$$

where  $h = \frac{b}{a}$  and  $K(t, 1) = \frac{(1+t)^2}{4t}$  is the Kontrovich constant,  $r = \min\{\nu, 1-\nu\}$ ,  $R = \max\{\nu, 1-\nu\}$  and  $r' = \min\{2r, 1-2r\}$ .

In addition, with the same notations as above, another type of the reverse of Young inequality using Kontrovich constant is as follows: [12]

$$a\nabla_\nu b - r(\sqrt{a} - \sqrt{b})^2 \leq K(\sqrt{h}, 2)^{R'} a\sharp_\nu b, \quad (1.3)$$

where  $R' = \max\{2r, 1-2r\}$ .

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Note that the  $K(t, 2) \geq 1$  for all  $t > 0$  and attains its minimum at  $t = 1$ . Also  $K(t, 2) = K(\frac{1}{t}, 2)$ .

Recently, Liao and Wu [11] obtained the following refinement of inequality (1.1) and (1.2):

$$a\nabla_\nu b \geq \nu(\sqrt{a} - \sqrt{b})^2 + r((ab)^{\frac{1}{4}} - \sqrt{a})^2 + K(h^{\frac{1}{4}}, 2)^{r_1} a\sharp_\nu b, \quad (1.4)$$

$$a\nabla_\nu b \leq (1 - \nu)(\sqrt{a} - \sqrt{b})^2 - r((ab)^{\frac{1}{4}} - \sqrt{b})^2 + K(h^{\frac{1}{4}}, 2)^{-r_1} a\sharp_\nu b,$$

for  $0 < \nu \leq \frac{1}{2}$ , and

$$a\nabla_\nu b \geq (1 - \nu)(\sqrt{a} - \sqrt{b})^2 + r((ab)^{\frac{1}{4}} - \sqrt{b})^2 + K(h^{\frac{1}{4}}, 2)^{r_1} a\sharp_\nu b, \quad (1.5)$$

$$a\nabla_\nu b \leq \nu(\sqrt{a} - \sqrt{b})^2 - r((ab)^{\frac{1}{4}} - \sqrt{a})^2 + K(h^{\frac{1}{4}}, 2)^{-r_1} a\sharp_\nu b,$$

for  $\frac{1}{2} < \nu < 1$ , where  $r = \min\{2(1 - \nu), 1 - 2(1 - \nu)\}$  and  $r_1 = \min\{2r, 1 - 2r\}$ .

For more related inequalities see [1, 9, 10, 13, 15].

These numerical inequalities, leads to similar operator inequalities. For this purpose, let  $\mathbb{B}(\mathbb{H})$  stand for the  $C^*$ -algebra of all bounded linear operators on a complex Hilbert space  $\mathbb{H}$ . An operator  $A \in \mathbb{B}(\mathbb{H})$  is called self-adjoint if  $A = A^*$ , positive (and is denoted by  $A \geq 0$ ) if  $A$  is self-adjoint with non-negative spectrum and strictly positive if  $A$  is an invertible positive operator.

If  $\mathbb{H}$  is finite dimensional, of dimension  $n$ , then we identify  $\mathbb{B}(\mathbb{H})$  with  $\mathbb{M}_n$  of all  $n \times n$  complex matrices. In this case, we use the terms positive semidefinite and positive definite matrices, instead of positive and strictly positive operators, respectively.

The partial order  $A \leq B$ , on the class of self-adjoint operators, means that  $B - A$  is a positive operator.

The weighted arithmetic and geometric mean for strictly positive operators  $A, B$ , is defined by

$$A\nabla_\nu B = (1 - \nu)A + \nu B, \quad A\sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}.$$

In addition, the Heinz mean of  $A$  and  $B$  is defined as

$$H_\nu(A, B) = \frac{A\sharp_\nu B + A\sharp_{1-\nu} B}{2}.$$

See [2, 4] for more information about these means.

Using the above notations, the operator versions of Young inequality, its refinements and its reverses are proved. For instance, we have the following refinement of (1.4) and (1.5) is obtained in [11]. The other inequalities are in similar way.

**Theorem 1.1.** [11] *Let  $A, B \in \mathbb{B}(\mathbb{H})$  be positive invertible operators and positive real numbers  $m, m', M, M'$  satisfy either  $0 < m'I \leq A \leq mI < MI \leq B \leq M'I$  or  $0 < m'I \leq B \leq mI < MI \leq A \leq M'I$ .*

(I) *If  $0 < \nu \leq \frac{1}{2}$ , then*

$$A\nabla_\nu B \geq 2\nu(A\nabla B - A\sharp B) + r(A\sharp B - 2A\sharp_{\frac{1}{4}} B + A) + K(h^{\frac{1}{4}}, 2)^{r_1} A\sharp_\nu B,$$

and

$$A\nabla_\nu B \leq 2(1 - \nu)(A\nabla B - A\sharp B) - r(A\sharp B - 2A\sharp_{\frac{3}{4}} B + B) + K(h^{\frac{1}{4}}, 2)^{-r_1} A\sharp_\nu B,$$

(II) If  $\frac{1}{2} < \nu < 1$ , then

$$A\nabla_\nu B \geq 2(1 - \nu)(A\nabla B - A\sharp B) + r(A\sharp B - 2A\sharp_{\frac{3}{4}}B + B) + K(h^{\frac{1}{4}}, 2)^{r_1} A\sharp_\nu B,$$

and

$$A\nabla_\nu B \leq 2\nu(A\nabla B - A\sharp B) - r(A\sharp B - 2A\sharp_{\frac{1}{4}}B + A) + K(h^{\frac{1}{4}}, 2)^{-r_1} A\sharp_\nu B,$$

where  $h = \frac{M}{m}$ ,  $r = \min\{\nu, 1 - \nu\}$  and  $r_1 = \min\{2r, 1 - 2r\}$ .

The main aim of this paper, is to state a generalization of these inequalities. First, we present some generalizations of numerical inequalities and base of them we prove some refined operator versions of Young inequality and its reverse. Also some inequalities for Hilbert-Schmidt norm of matrices are obtained.

Throughout, for  $0 \leq \nu \leq 1$ , the notations  $m_k = \lfloor 2^k \nu \rfloor$  is the largest integer not greater than  $2^k \nu$ ,  $r_0 = \min\{\nu, 1 - \nu\}$ ,  $r_k = \min\{2r_{k-1}, 1 - 2r_{k-1}\}$ , for  $k \geq 1$  and  $R_k = 1 - r_k$ .

## 2. NUMERICAL RESULTS

Our first theorem, states a refined version of Young inequality and its reverse.

**Theorem 2.1.** *Let  $a, b$  be two positive real numbers and  $\nu \in [0, 1]$ . Then*

$$\begin{aligned} K(h^{\frac{1}{2^n}}, 2)^{r_n} a\sharp_\nu b &\leq a\nabla_\nu b - \sum_{k=0}^{n-1} r_k \left[ \left( a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2 \\ &\leq K(h^{\frac{1}{2^n}}, 2)^{R_n} a\sharp_\nu b, \end{aligned} \quad (2.1)$$

where  $h = \frac{b}{a}$ .

In addition, if  $\nu = \frac{p}{2^t}$  for some  $p, t \in \mathbb{N}$  with  $t > 1$ , then

$$\begin{aligned} K(h^{\frac{1}{2^{t-1}}}, 2)^{r_{t-1}} a\sharp_\nu b &= a\nabla_\nu b - \sum_{k=0}^{t-2} r_k \left[ \left( a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2 \\ &= K(h^{\frac{1}{2^{t-1}}}, 2)^{R_{t-1}} a\sharp_\nu b, \end{aligned}$$

*Proof.* First, we prove the left hand of inequality (2.1), by induction. For  $n = 1$ , we get to the inequality (1.1). Let inequality (2.1) holds for  $n$ .

For  $0 < \nu \leq \frac{1}{2}$ , we have

$$\begin{aligned} a\nabla_\nu b - r_0(\sqrt{a} - \sqrt{b})^2 &= a\nabla_\nu b - \nu(\sqrt{a} - \sqrt{b})^2 \\ &= 2\nu\sqrt{ab} + (1 - 2\nu)a \\ &= a\nabla_{2\nu}\sqrt{ab} \end{aligned}$$

Applying inequality (2.1) for two positive numbers  $a$  and  $\sqrt{ab}$  and  $2\nu \in (0, 1]$ , we have

$$\begin{aligned} a\nabla_\nu b - r_0(\sqrt{a} - \sqrt{b})^2 &= a\nabla_{2\nu}\sqrt{ab} \\ &\geq K(\sqrt{h}^{\frac{1}{2^n}}, 2)^{r_{n+1}} a\sharp_{2\nu}\sqrt{ab} + \sum_{k=0}^{n-1} r_{k+1} \left[ \left( a^{1-\frac{m_{k+1}}{2^k}} (\sqrt{ab})^{\frac{m_{k+1}}{2^k}} \right)^{\frac{1}{2}} \right. \\ &\quad \left. - \left( a^{1-\frac{m_{k+1}+1}{2^k}} (\sqrt{ab})^{\frac{m_{k+1}+1}{2^k}} \right)^{\frac{1}{2}} \right]^2 \\ &= K(h^{\frac{1}{2^{n+1}}}, 2)^{r_{n+1}} a\sharp_\nu b + \sum_{k=1}^n r_k \left[ \left( a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2. \end{aligned}$$

For  $\frac{1}{2} < \nu < 1$ , we can apply the first part for  $1 - \nu$  and replace  $a$  and  $b$ . Note that  $\lfloor 2^k(1 - \nu) \rfloor = 2^k - \lfloor 2^k\nu \rfloor - 1$ , if  $2^k\nu$  is not integer. Thus, if  $2^k\nu$  is not integer for each  $k$ , the inequality follows.

Now, let  $\nu = \frac{p}{2^q}$  for some  $q > 1$  and odd number  $p$ . Since for each  $i < q$ , the coefficient  $r_i \leq \frac{1}{2}$  is of the form  $\frac{p_i}{2^{q-i}}$ , for some odd number  $q_i$ , it can be concluded that  $r_{q-1} = \frac{1}{2} = R_{q-1}$ . So the equality follows.

A similar argument, leads to the second inequality.  $\square$

Changing the elements  $a$  and  $b$  in inequality (2.1), we can state the following result for Heinz mean.

**Corollary 2.2.** *Let  $a, b$  be two positive real numbers and  $\nu \in (0, 1)$ . Then*

$$\begin{aligned} K(h^{\frac{1}{2^n}}, 2)^{r_n} H_\nu(a, b) &\leq a\nabla b - \sum_{k=0}^{n-1} r_k \left[ H_{\frac{m_k}{2^k}}(a, b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a, b) + H_{\frac{m_k+1}{2^k}}(a, b) \right] \\ &\leq K(h^{\frac{1}{2^n}}, 2)^{R_n} H_\nu(a, b), \end{aligned}$$

where  $h = \frac{b}{a}$ .

In the following theorem, we state another version of the reverse of Young inequality.

**Theorem 2.3.** *Let  $a, b$  be two positive real numbers and  $\nu \in (0, 1)$ . Then*

$$a\nabla_\nu b \leq K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + (\sqrt{a} - \sqrt{b})^2 - \sum_{k=0}^{n-1} r_k \left[ \left( a^{\frac{m_k}{2^k}} b^{1-\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( a^{\frac{m_k+1}{2^k}} b^{1-\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2, \quad (2.2)$$

where  $h = \frac{b}{a}$ .

*Proof.* Applying arithmetic-geometric mean inequality we have

$$K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + K(h^{\frac{1}{2^n}}, 2)^{r_n} b\sharp_\nu a \geq 2\sqrt{ab}.$$

Using this inequality and applying inequality (2.1), we have

$$(\sqrt{a} - \sqrt{b})^2 - a\nabla_\nu b = b\nabla_\nu a - 2\sqrt{ab} - K(h^{\frac{1}{2^n}}, 2)^{-r_n} a\sharp_\nu b + \sum_{k=0}^{n-1} r_k \left[ \left( a^{\frac{m_k}{2^k}} b^{1-\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( a^{\frac{m_k+1}{2^k}} b^{1-\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2.$$

So the result follows.  $\square$

**Corollary 2.4.** *Let  $a, b$  be two positive real numbers and  $\nu \in (0, 1)$ . Then*

$$a\nabla b \leq K(h^{\frac{1}{2^n}}, 2)^{-r_n} H_\nu(a, b) + (\sqrt{a} - \sqrt{b})^2 - \sum_{k=0}^{n-1} r_k \left[ H_{\frac{m_k}{2^k}}(a, b) - 2H_{\frac{2m_k+1}{2^{k+1}}}(a, b) + H_{\frac{m_k+1}{2^k}}(a, b) \right],$$

where  $h = \frac{b}{a}$ .

*Remark 2.5.* Replacing  $a$  and  $b$  by their squares in (2.1) and (2.2), respectively, we obtain

$$\begin{aligned} K(h^{\frac{1}{2^{n-1}}}, 2)^{r_n} a^2 \#_\nu b^2 &\leq a^2 \nabla_\nu b^2 - \sum_{k=0}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right]^2 \\ &\leq K(h^{\frac{1}{2^{n-1}}}, 2)^{R_n} a^2 \#_\nu b^2 \end{aligned} \quad (2.3)$$

and

$$a^2 \nabla_\nu b^2 \leq K(h^{\frac{1}{2^{n-1}}}, 2)^{-r_n} a^2 \#_\nu b^2 + (a - b)^2 - \sum_{k=0}^{n-1} r_k \left[ a^{\frac{m_k}{2^k}} b^{1-\frac{m_k}{2^k}} - a^{\frac{m_k+1}{2^k}} b^{1-\frac{m_k+1}{2^k}} \right]^2, \quad (2.4)$$

where  $h = \frac{b}{a}$ .

The following two theorems, are useful to prove a version of these inequalities for the Hilbert-Schmidt norm of matrices.

**Theorem 2.6.** *Let  $a, b$  be two positive real numbers and  $\nu \in (0, 1)$ . Then*

$$\begin{aligned} K(h^{\frac{1}{2^{n-1}}}, 2)^{r_n} (a \#_\nu b)^2 &\leq (a \nabla_\nu b)^2 - r_0^2 (a - b)^2 - \sum_{k=1}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right]^2 \\ &\leq K(h^{\frac{1}{2^{n-1}}}, 2)^{R_n} (a \#_\nu b)^2, \end{aligned} \quad (2.5)$$

where  $h = \frac{b}{a}$ .

*Proof.* By a simple calculation, we have  $(a \nabla_\nu b)^2 - r_0^2 (a - b)^2 = a^2 \nabla_\nu b^2 - r_0 (a - b)^2$ . Using (2.3), we have

$$\begin{aligned} K(h^{\frac{1}{2^{n-1}}}, 2)^{r_n} (a \#_\nu b)^2 &\leq a^2 \nabla_\nu b^2 - \sum_{k=0}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right]^2 \\ &= (a \nabla_\nu b)^2 - r_0^2 (a - b)^2 - \sum_{k=1}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right]^2 \\ &\leq K(h^{\frac{1}{2^{n-1}}}, 2)^{R_n} (a \#_\nu b)^2 \end{aligned}$$

□

**Theorem 2.7.** *Let  $a, b$  be two positive real numbers and  $\nu \in (0, 1)$ . Then*

$$(a \nabla_\nu b)^2 \leq K(h^{\frac{1}{2^{n-1}}}, 2)^{-r_n} (a \#_\nu b)^2 + R_0^2 (a - b)^2 - \sum_{k=1}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_k+1}{2^k}} b^{\frac{m_k+1}{2^k}} \right]^2, \quad (2.6)$$

where  $h = \frac{b}{a}$ .

*Proof.* We have

$$\begin{aligned}
& (a\nabla_\nu b)^2 - (1 - r_0)^2(a - b)^2 \\
&= a^2\nabla_\nu b^2 - (1 - r_0)(a - b)^2 \\
&\leq K(h^{\frac{1}{2^{n-1}}}, 2)^{-r_n}(a\sharp_\nu b)^2 + r_0(a - b)^2 - \sum_{k=0}^{n-1} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_{k+1}}{2^k}} b^{\frac{m_{k+1}}{2^k}} \right]^2 \\
&\quad \text{by inequality (2.4)} \\
&= K(h^{\frac{1}{2^{n-1}}}, 2)^{-r_n}(a\sharp_\nu b)^2 - \sum_{k=1}^{\infty} r_k \left[ a^{1-\frac{m_k}{2^k}} b^{\frac{m_k}{2^k}} - a^{1-\frac{m_{k+1}}{2^k}} b^{\frac{m_{k+1}}{2^k}} \right]^2.
\end{aligned}$$

□

### 3. RELATED OPERATOR INEQUALITIES

To state the operator versions of the inequalities obtained in section 2, we need the following lemma.

**Lemma 3.1.** [3] *Let  $X \in \mathbb{B}(\mathbb{H})$  be self-adjoint and let  $f$  and  $g$  be continuous real functions such that  $f(t) \geq g(t)$  for all  $t \in \sigma(X)$  (the spectrum of  $X$ ). Then  $f(X) \geq g(X)$ .*

Let  $X$  be a strictly positive operator. Then  $\sigma(X)$  is a compact subset of  $(0, +\infty)$ . We denote by  $m(X)$  and  $M(X)$  the minimum and the maximum of  $\sigma(X)$ .

Now, we give the first result in this section which is based on Theorem 2.1 and is a refinement of Theorem 3 in [11].

**Theorem 3.2.** *Let  $A, B \in \mathbb{B}(H)$  be two strictly positive operators with  $M(A) \leq m(B)$  and  $\nu \in (0, 1)$ .*

$$\begin{aligned}
K(h^{\frac{1}{2^n}}, 2)^{r_n} A\sharp_\nu B &\leq A\nabla_\nu B - \sum_{k=0}^{n-1} r_k \left[ A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_{k+1}}{2^k+1}} B + A\sharp_{\frac{m_{k+1}}{2^k}} B \right] \quad (3.1) \\
&\leq K(h^{\frac{1}{2^n}}, 2)^{R_n} A\sharp_\nu B,
\end{aligned}$$

where  $h = \frac{m(B)}{M(A)}$ .

*Proof.* Choosing  $a = 1$ , in Theorem 2.1, we have

$$1 - \nu + \nu b \geq K(b^{\frac{1}{2^n}}, 2)^{r_n} b^\nu + \sum_{k=0}^{n-1} r_k \left[ \left( b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( b^{\frac{m_{k+1}}{2^k}} \right)^{\frac{1}{2}} \right]^2,$$

for any  $b > 0$ .

If  $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ , then  $\sigma(X) \subseteq [h, +\infty)$ . Due to Kantorovich constant is increasing on  $[1, +\infty)$ , it follows that for all  $b \geq h$ ,

$$\begin{aligned} 1 - \nu + \nu b &\geq K(b^{\frac{1}{2^n}}, 2)^{r_n} b^\nu + \sum_{k=0}^{n-1} r_k \left[ \left( b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( b^{\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2 \\ &\geq K(h^{\frac{1}{2^n}}, 2)^{r_n} b^\nu + \sum_{k=0}^{n-1} r_k \left[ \left( b^{\frac{m_k}{2^k}} \right)^{\frac{1}{2}} - \left( b^{\frac{m_k+1}{2^k}} \right)^{\frac{1}{2}} \right]^2, \end{aligned}$$

According to Lemma 3.1, we get

$$(1 - \nu)I + \nu X \geq K(h^{\frac{1}{2^n}}, 2)^{r_n} X^\nu + \sum_{k=0}^{n-1} r_k \left[ X^{\frac{m_k}{2^k}} - 2X^{\frac{2m_k+1}{2^{k+1}}} + X^{\frac{m_k+1}{2^k}} \right]. \quad (3.2)$$

Multiplying both sides of (3.2) by  $A^{\frac{1}{2}}$ , we obtain

$$A\nabla_\nu B \geq K(h^{\frac{1}{2^n}}, 2)^{r_n} A\sharp_\nu B + \sum_{k=0}^{n-1} r_k \left[ A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}} B + A\sharp_{\frac{m_k+1}{2^k}} B \right].$$

This completes the proof of left hand of inequality (3.1), by the same way, we can prove the right hand.  $\square$

The following theorem is an operator version of Theorem 2.3 and is a refinement of Theorem 4 in [11].

**Theorem 3.3.** *Let  $A, B \in \mathbb{B}(\mathbb{H})$  be two positive invertible operators with  $M(A) \leq m(B)$  and  $\nu \in (0, 1)$ .*

$$A\nabla_\nu B \leq K(h^{\frac{1}{2^n}}, 2)^{-r_n} A\sharp_\nu B + (A - 2A\sharp B + B) - \sum_{k=0}^{n-1} r_k \left[ A\sharp_{\frac{m_k}{2^k}} B - 2A\sharp_{\frac{2m_k+1}{2^{k+1}}} B + A\sharp_{\frac{m_k+1}{2^k}} B \right],$$

where  $h = \frac{m(B)}{M(A)}$ .

*Proof.* By Lemma 2.3, using the same ideas as in the proof of Theorem 3.2, we can get this theorem.  $\square$

**Corollary 3.4.** *Let  $A, B \in \mathbb{B}(\mathbb{H})$  be two positive invertible operators with  $M(A) \leq m(B)$  and  $\nu \in (0, 1)$ . Then*

$$\begin{aligned} K(h^{\frac{1}{2^n}}, 2)^{r_n} H_\nu(A, B) &\leq A\nabla B - \sum_{k=0}^{n-1} r_k \left[ H_{\frac{m_k}{2^k}}(A, B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A, B) + H_{\frac{m_k+1}{2^k}}(A, B) \right] \\ &\leq K(h^{\frac{1}{2^n}}, 2)^{R_n} H_\nu(A, B) \end{aligned}$$

and

$$A\nabla B \leq K(h^{\frac{1}{2^n}}, 2)^{-r_n} H_\nu(A, B) + (A - 2A\sharp B + B) - \sum_{k=0}^{n-1} r_k \left[ H_{\frac{m_k}{2^k}}(A, B) - 2H_{\frac{2m_k+1}{2^{k+1}}}(A, B) + H_{\frac{m_k+1}{2^k}}(A, B) \right],$$

where  $h = \frac{m(B)}{M(A)}$ .

#### 4. MATRIX YOUNG AND REVERSE INEQUALITIES FOR THE HILBERT-SCHMIDT NORM

In this section, we present some inequalities for the Hilbert-Schmidt norm. It is known that every positive semidefinite matrix is unitarily diagonalizable. Thus for two positive semidefinite  $n \times n$  matrices  $A$  and  $B$ , there exist two unitary matrices  $U$  and  $V$  such that  $A = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*$  and  $B = V \text{diag}(\mu_1, \dots, \mu_n) V^*$ . Applying Theorem 2.6, we get the following theorem that is a refinement of the inequalities in [11, Theorem 5].

**Theorem 4.1.** *Suppose  $A, B, X \in \mathbb{M}_n$  such that  $A$  and  $B$  are two positive definite matrices and  $\nu \in (0, 1)$ . Let*

$$\underline{K}_t = \min \left\{ K \left( \left( \frac{\mu_j}{\lambda_i} \right)^{\frac{1}{2^t-1}}, 2 \right)^{r_t} : i, j = 1, 2, \dots, n \right\},$$

and

$$\overline{K}_t = \max \left\{ K \left( \left( \frac{\mu_j}{\lambda_i} \right)^{\frac{1}{2^t-1}}, 2 \right)^{R_t} : i, j = 1, 2, \dots, n \right\},$$

for all  $t \in \mathbb{N}$ . Then

$$\begin{aligned} \underline{K}_t \|A^{1-\nu} X B^\nu\|_2^2 &\leq \|(1-\nu)AX - \nu XB\|_2^2 - r_0^2 \|AX - XB\|_2^2 \\ &\quad - \sum_{k=1}^{t-1} r_k \|A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}} X B^{\frac{m_k+1}{2^k}}\|_2^2 \\ &\leq \overline{K}_t \|A^{1-\nu} X B^\nu\|_2^2. \end{aligned} \tag{4.1}$$

*Proof.* Let  $Y = U^* X V = (y_{ij})$ . Then

$$(1-\nu)AX - \nu XB = U[(1-\nu)\lambda_i + \nu\mu_j] \circ Y V^*,$$

$$AX - XB = U[(\lambda_i - \mu_j) \circ Y] V^*$$

$$A^{1-\nu} X B^\nu = U[(\lambda_i^{1-\nu} \mu_j^\nu) \circ Y] V^*$$

and

$$A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_k+1}{2^k}} X B^{\frac{m_k+1}{2^k}} = U[(\lambda_i^{1-\frac{m_k}{2^k}} \mu_j^{\frac{m_k}{2^k}} - \lambda_i^{1-\frac{m_k+1}{2^k}} \mu_j^{\frac{m_k+1}{2^k}}) \circ Y] V^*.$$



Utilizing the unitarily invariant property of  $\|\cdot\|_2$  and Theorem 2.6, we have

$$\begin{aligned}
& \|(1-\nu)AX - \nu XB\|_2^2 \\
&= \sum_{i,j=1}^n ((1-\nu)\lambda_i + \nu\mu_j)^2 |y_{ij}|^2 \\
&\geq \sum_{i,j=1}^n \left\{ K \left( \left( \frac{\mu_j}{\lambda_i} \right)^{\frac{1}{2^{t-1}}}, 2 \right)^{r_t} (\lambda_i^{1-\nu} \mu_j^\nu)^2 + r_0^2 (\lambda_i - \mu_j)^2 + \sum_{k=1}^{t-1} r_k (\lambda_i^{1-\frac{m_k}{2^k}} \mu_j^{\frac{m_k}{2^k}} - \lambda_i^{1-\frac{m_{k+1}}{2^k}} \mu_j^{\frac{m_{k+1}}{2^k}})^2 \right\} |y_{ij}|^2 \\
&= \sum_{i,j=1}^n K \left( \left( \frac{\mu_j}{\lambda_i} \right)^{\frac{1}{2^{t-1}}}, 2 \right)^{r_t} (\lambda_i^{1-\nu} \mu_j^\nu)^2 |y_{ij}|^2 + \sum_{i,j=1}^n r_0^2 (\lambda_i - \mu_j)^2 |y_{ij}|^2 \\
&\quad + \sum_{i,j=1}^n \left\{ \sum_{k=1}^{t-1} r_k (\lambda_i^{1-\frac{m_k}{2^k}} \mu_j^{\frac{m_k}{2^k}} - \lambda_i^{1-\frac{m_{k+1}}{2^k}} \mu_j^{\frac{m_{k+1}}{2^k}})^2 |y_{ij}|^2 \right\} \\
&\geq \underline{K}_t \sum_{i,j=1}^n (\lambda_i^{1-\nu} \mu_j^\nu)^2 |y_{ij}|^2 + \sum_{i,j=1}^n r_0^2 (\lambda_i - \mu_j)^2 |y_{ij}|^2 \\
&\quad + \sum_{k=1}^{t-1} \left\{ \sum_{i,j=1}^n r_k (\lambda_i^{1-\frac{m_k}{2^k}} \mu_j^{\frac{m_k}{2^k}} - \lambda_i^{1-\frac{m_{k+1}}{2^k}} \mu_j^{\frac{m_{k+1}}{2^k}})^2 |y_{ij}|^2 \right\} \\
&= \underline{K}_t \|A^{1-\nu} X B^\nu\|_2^2 + r_0^2 \|AX - XB\|_2^2 \\
&\quad + \sum_{k=1}^{t-1} r_k \|A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_{k+1}}{2^k}} X B^{\frac{m_{k+1}}{2^k}}\|_2^2.
\end{aligned}$$

This complete the proof of the left side of (4.1). By the same ideas, we can prove the right side.  $\square$

**Theorem 4.2.** Suppose  $A, B, X \in \mathbb{M}_n$  such that  $A$  and  $B$  are positive definite matrices and  $\nu \in (0, 1)$ . Let

$$\underline{K}_t = \min \left\{ K \left( \left( \frac{\mu_j}{\lambda_i} \right)^{\frac{1}{2^{t-1}}}, 2 \right)^{r_t} : i, j = 1, 2, \dots, n \right\}$$

. Then

$$\begin{aligned}
\|(1-\nu)AX - \nu XB\|_2^2 &\leq \underline{K}_t^{-1} \|A^{1-\nu} X B^\nu\|_2^2 + R_0^2 \|AX - XB\|_2^2 \\
&\quad - \sum_{k=1}^{\infty} r_k \|A^{1-\frac{m_k}{2^k}} X B^{\frac{m_k}{2^k}} - A^{1-\frac{m_{k+1}}{2^k}} X B^{\frac{m_{k+1}}{2^k}}\|_2^2.
\end{aligned}$$

*Proof.* By Theorem 2.7, using the same idea as in the proof of Theorem 4.1, we can obtain the desired result.  $\square$

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